# What is the physical significance of negative frequencies?

This has been one of the holes in my cheddar cheese block of understanding DSP, so what is the physical interpretation of having a negative frequency?

If you have a physical tone at some frequency and it is DFT'd, you get a result in both the positive and negative frequencies - why and how does this occur? What does it mean?

Edit: Oct 18th 2011. I have provided my own answer, but expanded the question to include the roots of why negative frequencies MUST exist.

frequency

edited Oct 19 '11 at 9:04

asked Oct 15 '11 at 19:41
Mohammad
4,170 ● 2 ● 11 ● 37

electronics.stackexchange.com/questions/15539/... – endolith Oct 18 '11 at 0:59

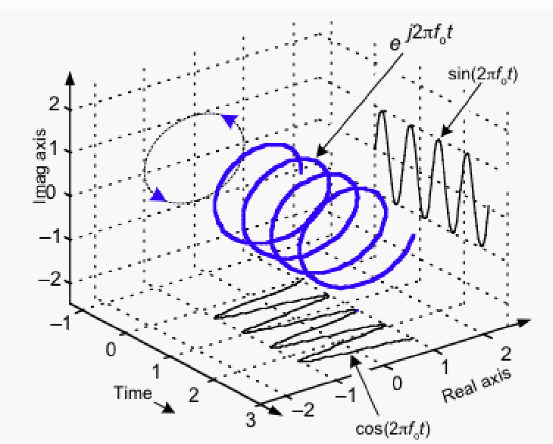
Thanks endolith, would it be possible to cross link this page to them? I have provided an answer to my own question and would like to share it with that group too. I dont seem to have access to that area... – Mohammad Oct 19 '11 at 9:06

## 7 Answers

Negative frequency doesn't make much sense for sinusoids, but the Fourier transform doesn't break up a signal into sinusoids, it breaks it up into complex exponentials:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

These are actually spirals, spinning around in the complex plane:




Spirals can be either left-handed or right-handed (rotating clockwise or counterclockwise), which is where the concept of negative frequency comes from. You can also think of it as the phase angle going forward or backward in time.

In the case of real signals, there are always *two* equal-amplitude complex exponentials, rotating in opposite directions, so that their real parts combine and imaginary parts cancel out, leaving only a real sinusoid as the result. This is why the spectrum of a sine wave always has 2 spikes, one positive frequency and one negative. Depending on the phase of the two spirals, they could cancel out leaving a purely real sine wave or a real cosine wave or a purely imaginary sine wave, etc.

The negative and positive frequency components *are* both necessary to produce the real signal, but if you already know that it's a real signal, the other side of the spectrum doesn't provide any extra information, so it's often handwaved and ignored. For the general case of complex signals, you need to know both sides of the frequency spectrum.

edited Aug 27 '13 at 18:08

answered Oct 17 '11 at 14:47

endolith

5,194 ● 10 ● 41

1 I like that description; I think the diagram explains it well. – Jason R Oct 18 '11 at 13:16

@endolith Nice post - I have seen this from Lyons book btw. It would seem to me that the actual 'starting' point for all oscillations is in the complex domain, and that it just so happens that we can only measure realistic oscillations that occur on the real-axis. So when a physical wave is measured, it is taken BACK into the complex domain, which is where we see its clockwise and counter-clock wise components. Which is funny because 'real' signals end up being 'twice as complicated' as complex signals... – Mohammad Oct 19 '11 at 3:43

@Mohammad: I don't know about complex exponentials being more "fundamental" than sinusoids in general, though they are in the case of the Fourier transform. You can produce complex exponentials by adding sinusoids, and sinusoids by adding complex exponentials. They're all just functions. Sinusoids are generally derived from circles, which may be something in the complex plane, or may just be the height of a dot on a spinning wheel. – endolith Oct 19 '11 at 13:41

@endolith Right. I have expanded on that some in my post. Either way great post (and thanks for the cross link). Have an upvote! :-) – Mohammad Oct 20 '11 at 0:54

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Let's say you had a spinning wheel. How would you describe how fast it is spinning? You'd probably say it's spinning at `X` revolutions per minute (rpm). Now how do you convey in what *direction* it's spinning with this number? It's the same `X` rpm if it's spinning clockwise or anti-clockwise. So you scratch your head and say oh well, here's a smart idea: I'll use the convention of `+X` to indicate that it's spinning clockwise and `-X` for anti-clockwise. Voila! You've invented negative rpms!

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Negative frequency is no different from the above simple example. A simple mathematical explanation of how the negative frequency pops up can be seen from the Fourier transforms of pure tone sinusoids.

Consider the Fourier transform pair of a complex sinusoid: $e^{\jmath\omega_0 t} \longleftrightarrow \delta(\omega+\omega_0)$ (ignoring constant multiplier terms). For a pure sinusoid (real), we have from Euler's relation:

$$\cos(\omega_0 t) = \frac{e^{\jmath\omega_0 t} + e^{-\jmath\omega_0 t}}{2}$$

and hence, its Fourier transform pair (again, ignoring constant multipliers):

$$\cos(\omega_0 t) \longleftrightarrow \delta(\omega+\omega_0) + \delta(\omega-\omega_0)$$

You can see that it has two frequencies: a positive one at $\omega_0$ and a negative one at $-\omega_0$ by definition! The complex sinusoid of $ae^{\jmath\omega_0 t}$ is widely used because it is incredibly useful in simplifying our mathematical calculations. However, it has only one frequency and a real sinusoid actually has two.

answered Oct 15 '11 at 23:26

Lorem Ipsum

3,593 ● 2 ● 19 ● 33

2 thanks for the answer - I understand the math - and this is something basic I know of, but it doesnt give us information on the physical meaning... Going on your spinning example though - ok, so the sign of the frequency conveys the 'direction' of the change in phase. Fair enough, but still, why does a sinusoid have 'two' frequencies, one positive and one negative? Is it because the fourier transform is 'time agnostic', and so you can look at a real sinusoid in the real direction of time, get your +ve, and look at the same wave backwards in time and get your -ve? Thanks. – Mohammad Oct 16 '11 at 3:16

4 I'm not sure that there is a concrete answer to your confusion. Content at negative frequencies is a consequence of the definition of the Fourier transform and doesn't directly have a physical meaning. The Fourier transform isn't inherently a "physical" operation, so it doesn't have to. A sinusoid's frequency is the time derivative of phase,

nothing more. Negative frequencies are just a mathematical artifact that some people get hung up on, similar to the use of "imaginary" parts of complex numbers. They are analysis tools used for modeling, not necessarily existing in the physical world. – Jason R Oct 16 '11 at 4:46

1 @Mohammad I agree with Jason here. At some point, trying to construct a "physical" explanation for the sake of it can only make things worse. I'm not sure I can explain any better... – Lorem Ipsum Oct 16 '11 at 4:56

...Umm, the hallmark of science is to attach physical meaning to reality and the math we use to describe it. It doesnt make anything 'worse'. I got the same response from professors talking about "complex band pass signals", until we realized imaginary components are just the quadrature of your signal, and the real components are the in-phase. That is a physical interpretation of complex signals. So I am asking, what is the physical interpretation of the negative frequencies? You interpret the positive side to tell you your rate of oscillation, what does the -ve tell you? – Mohammad Oct 16 '11 at 5:35

2 A possible explanation is that from the point of the Fourier transform, a real sinusoid is "really" the sum of two complex sinusoids spinning in opposite directions. Using the wheel analogy: Picture two wheels at the origin of a coordinate system, spinning at the same speed but in opposite directions, with a pin on each that starts at (1,0). Now add the coordinates of both pins: y will always be 0, and x will be a real sinusoid. – Sebastian Reichelt Oct 16 '11 at 10:51

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Currently, my viewpoint (it is subject to change) is the following

For sinusoidal repetition only positive frequencies makes sense. The physical interpretation is clear. For complex exponential repetition both positive and negative frequencies makes sense. It may be possible to attach a physical interpretation to negative frequency. That physical interpretation of negative frequency has to do with direction of repetition.

The definition of frequency as provided on wiki is: "Frequency is the number of occurrences of a repeating event per unit time"

If sticking to this definition negative frequency does not make sense and therefore has no physical interpretation. However, this definition of frequency is not thorough for complex exponential repetition which can also have direction.

Negative frequencies are used all the time when doing signal or system analysis. The fundamental reason for this being the euler formula

$$e^{j\omega n} = cos(\omega n) + j\,sin(\omega n)$$

and the fact that complex exponentials are eigenfunctions of LTI systems.

The sinusoidal repetition is normally of interest and the the complex exponential repetition is often used to obtain the sinusoidal repetition indirectly. That the two are related can be easily seen by considering the Fourier representation written using complex exponentials e.g.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \quad X(e^{j\omega})e^{j\omega n}$$

However, this is equivalent to

$$x[n] = \frac{1}{2\pi}\int_{0}^{\pi} d\omega \ \ [a(\omega)cos(\omega n) + b(\omega)sin(\omega n)] = \frac{1}{2\pi}\int_{0}^{\pi} d\omega \ \ \alpha(\omega)sin(\omega n + \phi(\omega))]$$

So instead of considering a postive 'sinusoidal frequency axis', a negative and positive 'complex exponential frequency axis' is considered. On the 'complex exponential frequency axis' it is well known that the negative frequency part is redundant and only the positive 'complex exponential frequency axis' is considered. In making this step implicitly we know that the frequency axis represents complex exponential repetition and not sinusoidal repetition.

The complex exponential repetition is a circular rotation in the complex plane. In order to create a sinusoidal repetition it takes two complex exponential repetitions, one repetition clock-wise and one repetition counter clock-wise. If a physical device is constructed that produces a sinusoidal repetition inspired by how the sinusoidal repetition is created in the complex plane, that is, by two physically rotating devices that rotates in opposite directions, one of the rotating devices can be said to have a negative frequency and thereby the negative frequency has a physical interpretation.

answered Oct 17 '11 at 10:06

niaren
1,362 5 9

I like your explanation... slowly a picture is emerging, see my answer / edit-to-question. – Mohammad Oct 19 '11 at 3:23

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In many common applications negative frequencies have no direct physical meaning at all. Consider a case where there is an input and an output voltage in some electrical circuit with resistors, capacitors, and inductors. There is simply a real input voltage with one frequency and there is a single output voltage with the same frequency but different amplitude and phase.

The ONLY reason why you would consider complex signals, complex Fourier Transforms and phasor math at this point is mathematically convenience. You could do it just as well with entirely real math, it would just be a lot harder.

There are different types of time/frequency transforms. The Fourier Transform uses a complex exponential as its basis function and applied to a single real-valued sine wave happens to produces a two valued results which is interpreted as positive and negative frequency. There are other transforms (like the Discrete Cosine Transform) which would not produce any negative frequencies at all. Again, it's a matter of mathematical convenience; the Fourier Transform is often the quickest and most efficient way to solve a specific problem.

answered Oct 17 '11 at 15:40
Hilmar
7,423 ● 7 ● 19

I agree, it is certainly a lot more convenient to work in the complex domain - the 'issue' creeps up because some individuals claim that there is no physical meaning to negative frequencies, yet somehow they possess energy in the frequency domain. Well, if they aren't 'really there', then where is this energy? – Mohammad Oct 19 '11 at 3:20

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You should study the Fourier transform or series to understand the negative frequency. Indeed Fourier showed that we can show all of waves using some sinusoids. Each sinusoid can be shown with two peaks at the frequency of this wave one in positive side and one in negative. So the theoretical reason is clear. But for the physical reason, I always see that people say negative frequency has just mathematical meaning. But I guess a physical interpretation that I'm not pretty sure; When you study the circular motion as the principal of discussions about the waves, the direction of speed of the movement on the half-circle is inverse of the another half. This can be the reason why we have two peaks in both sides of the frequency domain for each sine wave.

answered Oct 15 '11 at 22:58
Hossein
131 ● 7

Hossein, yes, I agree it had be confused for a while. I am waiting on yoda for his feedback, but if it is just simply the sign of the derivative of the phase, then I see a linguistic problem - perhaps the source of confusion with the many other folks I have talked to about this as well. The physical meaning of a 'frequency' is 'the rate of oscillation' of something, meaning is has to be positive. This is where I think the definitions differ from that in physics. – Mohammad Oct 16 '11 at 3:48

Please look at the page en.wikipedia.org/wiki/Circular_motion; $w = 2 * \pi/T$ and $f = 1/T$ so f and w have direct relation. In each wave, the direction of speed is changed to have a complete oscillation. We always should take care that a real wave needs two rates to be a complete one. In practice when you work with spectrum analyzer, you've just positive part because it is sufficient. The negative part is quite meaningful because in case of the shift, you can see this negative part on spectrum analyzer that shows just positive parts. – Hossein Oct 16 '11 at 14:48

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What is the meaning of negative distance? One possibility is that it's for continuity, so you don't have to flip planet Earth upside down every time you walk across the equator, and want to plot your position North with a continuous 1st derivative.

Same with frequency, when one might do such things as FM modulation with a modulation wider than the carrier frequency. How would you plot that?

answered Oct 16 '11 at 7:30
hotpaw2
11.8k ● 1 ● 11 ● 33

See my new answer / edit to question – Mohammad Oct 19 '11 at 3:21

This has turned out to be quite the hot topic.

After reading the rich multitude of good and diverse opinions and interpretations and letting the issue simmer in my head for sometime, I believe I have a physical interpretation of the phenomenon of negative frequencies. And I believe the key interpretation here is that fourier is blind to time. Expaning on this further:

There has been a lot of talk about the 'direction' of the frequency, and thus how it can be +ve or -ve. While the overarching insights of the authors saying this is not lost, this statement is nontheless inconsistent with the definition of temporal frequency, so first we must define our terms very carefully. For example:

- *Distance* is a scalar (can only ever be +ve), while *displacement* is a vector. (ie, has direction, can be +ve or -ve to illustrate heading).
- *Speed* is a scalar (can only be +ve), while *velocity* is a vector. (ie, again, has direction, and can be +ve or -ve).

Thus by the same tokens,

- *Temporal Frequency* is a scalar, (can only be +ve)! Frequency is defined as number of cycles per unit time. If this is the accepted definition, we **cannot** simply claim that it is going in 'a different direction'. Its a scalar after-all. Instead, we must define a new term - the vector equivalent of frequency. Perhaps 'angular frequency' would be the right terminology here, and indeed, that is precisely what a *digital* frequency measures.

Now all the sudden we are in the business of measuring number of *rotations* per unit time, (a vector quantity that can have direction), VS just the number of repititions of some physical oscillation.

Thus when we are asking about the physical interpretation of negative frequencies, we are also implicitly asking about **how** the scalar and very real measures of number of oscillations per unit time of some physical phenomenon like waves on a beach, sinusoidal AC current over a wire, *map* to this angular-frequency that now all the sudden happens to have direction, either clockwise or counterclockwise.

From here, to arrive at a physical interpretation of negative frequencies two facts need to be heeded. The first one is that as Fourier pointed out, an oscillatory real tone with *scalar temporal frequency, f*, can be constructed by adding two oscillatory complex tones, with *vector angular frequencies, +w and -w* together.

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Thats great, but so what? Well, the complex tones are rotating in directions opposite to each other. (See also Sebastian's comment). But what is the significance of the 'directions' here that give our angular frequencies their vector status? What physical quantity is being reflected in the direction of rotation? The answer is time. In the first complex tone, time is travelling in the +ve direction, and in the second complex tone, time is travelling in the -ve direction. Time is going backwards.

Keeping this in mind and taking a quick diversion to recall that temporal frequency is the first derivative of phase with respect to time, (simply the change of phase over time), everything begins to fall into place:

**The physical interpretation of negative frequencies is as follows:**

My first realization was that *fourier is time-agnostic*. That is, if you think about it, there is nothing in fourier analysis or the transform itself that can tell you what the 'direction' of time is. Now, imagine a physically oscillating system (ie a real sinusoid from say, a current over a wire) that is oscillating at some *scalar temporal-frequency, f*.

Imagine 'looking' down this wave, in the forwards direction of time as it progresses. Now imagine calculating its difference in phase at every point in time you progress further. This will give you your scalar temporal frequency, and your frquency is positive. So far so good.

But wait a minute - if fourier is blind to time, then why should it only consider your wave in the 'forward' time direction? There is nothing special about that direction in time. Thus by symmetry, the other direction of time must also be considered. Thus now imagine 'looking' up at the same wave, (ie, *backwards* in time), and also performing the same delta-phase calculation. Since time is going backwards now, and your frequency is change-of-phase/(negative time), your frequency will now be negative!

What Fourier is really saying, is that this signal has energy if played forward in time at frequency bin f, but ALSO has energy if played *backwards* in time albeit at frequency bin -f. In a sense it MUST say this because fourier has no way of 'knowing' what the 'true' direction of time is!

So how does fourier capture this? Well, in order to show the *direction* of time, a rotation of some sort *must* be employed such that a clockwise roation dealinates 'looking' at the signal in the forward arrow of time, and a counterclockwise roation dealinates 'looking' at the signal as if time was going backwards. The scalar temporal frequency we are all familiar with should now be equal to the (scaled) absolute value of our vector angular frequency. But how can a point signifying the displacement of a sinusoid wave arrive at its starting point after one cycle yet simultaneously rotate around a circle and maintain a manifestation of the temporal frequency it signifies? Only if the major axes of that circle are composed of measuring displacement of this point relative to the original sinusoid, and a sinusoid off by 90 degrees. (This is exactly how fourier gets his sine and cosine bases the you project against every time you perform a DFT!). And finally, how do we keep those axes seperate? The 'j' guarantees that the magnitude on each axis is always independant of the magnitude on the other, since real and imaginary numbers cannot be added to yield a new number in either domain. (But this is just a side note).

Thus in summary:

The fourier transform is time-agnostic. It cannot tell the direction of time. This is at the heart of negative frequencies. Since frequency = phase-change/time, anytime you take the DFT of a signal, fourier is saying that if time was going forwards, your energy is located on the +ve frequency axis, but if your time was going backwards, your energy is located on the -ve frequency axis.

As our universe has shown before, it is precisely because Fourier does not know the direction of time, that both sides of the DFT *must* be symmetric, and why the existence of negative frequencies are necessary and in fact very real indeed.

edited Oct 20 '11 at 1:01

answered Oct 19 '11 at 8:36
Mohammad
4,170 ● 2 ● 11 ● 37

1 I think you're reading a bit too much into this in an attempt to justify an answer that you already have decided upon. The roots of "negative" frequencies have been pointed out in other answers. The Fourier transform uses complex exponentials as its basis functions. Their complex nature makes it possible to discriminate the sign of the exponential's frequency as time increases. Complex exponentials are of interest because they are eigenfunctions of linear time-invariant systems. That makes the FT very useful as an signal and systems analysis tool. – Jason R Oct 19 '11 at 13:42

3 The negative frequencies that exist in the complex-exponential decomposition of signals are part of the package that comes along with using the Fourier transform. There is no need to come up with a complicated, qualitative explanation for what they must mean. – Jason R Oct 19 '11 at 13:43

1 Also, I think your first bullet might be in error; I've always heard *distance* referred to as a scalar, while *displacement* is a vector quantity. – Jason R Oct 19 '11 at 13:44

1 Also, in addition to what Jason said, I really fail to see the "physical" aspect in this answer, that *you* said was lacking in all the others... – Lorem Ipsum Oct 19 '11 at 14:08

@JasonR I know my post is long, but please *do* try to read my post (fully) before commenting on it in the future. When you do you will see that it is not complicated but fits nicely with what we know so far. You will see how my explanation is actually *derived* and *built* from all the prior answers and my research into the literature. – Mohammad Oct 20 '11 at 0:41

@JasonR Going back to the subject at hand, I have shown how negative frequencies have to be interpreted in the context of interpreted of time flowing backwards, with one main reason being that frequency is nothing but the rate of change of phase over time, and the only way for this to occur is for time to flow backwards given a constant delta-phase. – Mohammad Oct 20 '11 at 0:42

@JasonR A sinusoid is the sum of two phasors, one of which rotates clockwise, the other one of which rotates counter clockwise. One helix was drawn earlier as it evolved in time. If the other helix it is being added to is rotating in the opposite direction, this means it is evolving backwards in time. This is where the 'negative' comes from. – Mohammad Oct 20 '11 at 0:42

@yoda See last comment for physical aspect. Physically, the signal is 'looked at' as if time was reversed, the first derivative of its phase changes computed with time flowing in the reverse direction. When it is projected onto the bases of fourier, this fact must be reflected this fourier is blind to time. Also, please provide a quote on what you claim I said. – Mohammad Oct 20 '11 at 0:45